Decentralized Pricing*

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Abstract

The paper studies the conditions under which an intermediary can decentralize the pricing decisions of a transaction to privately informed parties. The analysis shows that decentralized pricing is both necessary and sufficient for ex post incentive compatibility if the parties have negatively interdependent transaction values (as is often the case in transactions between buyers and sellers: an increase in the quality of the good makes purchasing it more attractive, but increases the seller's opportunity cost). On the contrary, with positive interdependence, we obtain a negative result. The results provide new insights into robust trading mechanisms, the equivalence between Bayesian and dominant strategy implementation, tax incidence, and pricing in two-sided markets.

Keywords: bilateral trade, mechanism design, ex post equilibrium, interdependent values, two-sided markets, double auction, tax incidence.

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1 Introduction

Prices are messages: they contain information (Hayek, 1945). But they also provide incentives (Hurwicz, 1972), which begs the question: when do prices create the correct incentives *and* capture all relevant information? For example, the potential buyer of a used car may inquire of the seller whether or not the car has been in an accident. However, that information may already be included in the price asked by the seller, as it would affect the seller's opportunity cost of trade.

This paper takes a mechanism design approach to study this question. It considers the problem of an intermediary organizing a transaction between two parties; each party privately observes a signal about the value of the transaction and the intermediary seeks to implement an allocation, which maps the parties' information into a trading decision and transfers. To achieve this, the intermediary chooses a revelation mechanism, consisting of message spaces for the parties and of a decision rule. The solution concept is ex post equilibrium.

The key insight from the analysis is that prices are sufficiently informative when the values that the parties attach to the transaction are negatively interdependent, as is often the case in bilateral trade. For instance, bad news about the quality of a good makes the seller eager to sell but reduces the buyers' willingness to buy. When instead the transaction values are positively interdependent (as is typical in matching markets, for example), then prices do not suffice to capture all of the relevant information.

The analysis focuses on ex post implementation. A strategy profile of an incomplete information game is an ex post equilibrium if every action profile is a Nash equilibrium for every possible state of information (Bergemann and Morris, 2008). An allocation is weakly ex post implementable if there exists a mechanism with an ex post equilibrium that delivers it. If all ex post equilibria give the allocation, then it is strongly ex post implementable.

Ex post equilibria are Bayesian equilibria with no regret: once the resources have been allocated, no player would like to change their action even if all private information were to become public. As opposed to other Bayesian equilibria, ex post equilibria are thus robust to assumptions about the informational structure, so the results hold no matter what beliefs players have about each other. The concept is weaker than dominant strategies, however, as the equilibrium strategies need not best respond to actions outside the equilibrium.

Weak ex post implementation is more demanding than weak Bayesian implementation, but less demanding than implementation in weakly dominant strategies. When values are interdependent, ex post equilibrium is much more tractable than dominant strategies, which allow the implementation of very little, if anything (Williams and Radner, 1988). But when values are private, implementation in weakly dominant strategies is equivalent to weak ex post implementation.

For strong implementation the ranking is not clear, as undesired equilibria must be eliminated. Bergemann and Morris (2008) derive an ex post monotonicity condition that is necessary for strong ex post implementation, and also sufficient in a wide class of environments with at least three agents. They also identify settings in which the direct mechanism is sufficient for strong ex post implementation.

Assuming one-dimensional signals and negatively interdependent values, the first result is that an allocation is weakly ex post implementable if and only if it can be weakly ex post implemented by a price mechanism, in which each party sets a price that the other must pay to the intermediary in order to complete the transaction and a fixed fee. These prices can be negative. The intermediary makes a trade decision based on the transaction prices.

The intuition for this result is quite simple. If a seller with good news ends up selling the good, then sellers with bad news, having lower opportunity costs, will also sell: they benefit more from the transaction and can always mimic the seller with good news. For the buyer to have the correct incentives to purchase under bad news, the price must then decrease, as the buyer benefits less from the transaction. It then follows that, for every price charged to the buyer, there exists a unique signal observed by the seller. Therefore, the price charged to the buyer conveys all of the information observed by the seller. The same argument applies to the seller's price and the buyer's signal. Without negative interdependence the argument – which relies on monotonicity – breaks down and different signals can be associated with the same price, but still lead to different trades.

This is, in particular, the case with private values: it is possible to construct a weakly dominant strategy implementable (or equivalently weakly ex post implementable) allocation that cannot be implemented by any price mechanism. However, the implementation result does extend to allocations with continuous marginal types, which is often the case in situations of interest.

The paper also provides a result for strong ex post implementation. With negatively interdependent and additively separable values, the allocation is strongly ex post implementable by the direct mechanism if and only if it is so by a price mechanism. This result generalizes to arbitrary signal spaces. By further assuming connected signal spaces and continuous transaction values, the result on weak ex post implementation extends to multi-dimensional signals, as long as the allocation can be written as a function of the values obtained from the transaction and the values are additively separable.

Related literature. Milgrom and Weber (1982) introduce a model of auctions with informational externalities and Crémer and McLean (1985) provide sufficient conditions for extracting all the gains from trade in ex post equilibrium when demands are interdependent. For multi-dimensional signals, Dasgupta and Maskin (2000) show that no auction is generally efficient, and Jehiel et al. (2006) show that the only deterministic allocations that are generically ex post implementable, are constants. A similar result is obtained by Jehiel and Moldovanu (2001) on efficient Bayesian implementation.

Instead of studying *what* can be implemented ex post, this paper shows *how* it can be achieved. For the case of private values, the paper brings together three seminal papers from the 1980s: Myerson and Satterthwaite (1983), Chatterjee and Samuelson (1983) and Hagerty and Rogerson (1987). Chatterjee and Samuelson (1983) consider bilateral bargaining without an intermediary: the buyer and the seller simultaneously submit price offers, which determine whether the good is sold and at what price. In the current paper, the parties also submit price offers, but an intermediary acts as a budget breaker and takes the bid-ask spread as a fee for executing the transaction.

Myerson and Satterthwaite (1983) show the general impossibility of ex post efficient, budget balanced and individually rational, Bayesian incentive compatible trading mechanisms. They also derive mechanisms that maximize the expected revenue and expected welfare under budget balance. Gershkov et al. (2013) show that, with linear utilities and private values, for any Bayesian incentive compatible mechanism, there exists a dominant strategy incentive compatible counterpart, which is equivalent in that it gives the same expected revenue and interim expected payoffs. In a simple environment with two parties, this paper adds to this by showing that these mechanisms are in fact price mechanisms.

Hagerty and Rogerson (1987) consider the problem of designing an ex post incentive compatible, ex post individually rational and budget balanced trading institution.[1] Assuming private values, they show that posted-price mechanisms are essentially the only mechanisms that satisfy all three constraints. In this paper we show that, if the values are negatively interdependent, then this set is essentially empty.

A shortcoming of direct revelation mechanisms is that it is difficult to make general

[1]Hagerty and Rogerson (1987) consider dominant strategy incentive compatibility, which in their environment is equivalent to ex post incentive compatibility.

statements about tax incidence. The implementation results derived in this paper can be applied to overcome this problem in bilateral trade contexts, where tax incidence is often an important issue. Following Weyl and Fabinger (2013), who extend the principles of tax incidence obtained for perfect competition to models of imperfect competition, we extend them to ex post incentive compatible trading mechanisms.

Belleflamme and Toulemonde (2018) study tax incidence in two-sided markets. When a tax is levied on one side of the market, they show that this side fully bears it, whereas the other side and the platforms are not affected by the tax. We instead show that, under incomplete information, regardless of who pays the tax, its burden is split between the intermediary and the two sides of the market, as both sides pay a higher transaction price to the intermediary.

Finally, the paper also relates to the literature on two-sided markets, initiated by Caillaud and Jullien (2003), Rochet and Tirole (2006) and Armstrong (2006), then followed by Weyl (2010) among the others. Revisiting the canonical model of Rochet and Tirole (2003) we derive the optimal price mechanism, which performs better than linear prices and still resolves the chicken and egg problem.

The paper is organized as follows. The next section introduces the model and defines price mechanisms. Section 3 establishes the main results for weak ex post implementation in the case of one-dimensional signal spaces. Section 4 extends these results to multi-dimensional signals and presents the main result for strong ex post implementation, which holds for arbitrary signal spaces. Section 5 is devoted to applications and Section 6 concludes.

2 Model

An intermediary brings together two parties, called X and Y. If the parties complete a transaction, each $i = X, Y$ obtains a value $v_i(x, y) \in \mathbb{R}$, where $x \in \mathscr{X}$ is privately observed by X and $y \in \mathscr{Y}$ is privately observed by Y. We refer to (x, y) as types or signals about the value of the transaction. The sets $\mathscr{X}$ and $\mathscr{Y}$ are nonempty. The intermediary is not informed about the value of the transaction.

Both parties have additively separable utility in money and the transaction. Letting $t_i \in \mathbb{R}$ denote the transfer paid by party i to the intermediary and $q \in \{0, 1\}$ indicate whether the transaction takes place or not, i obtains utility

$$u_i(q, t_i; x, y) = v_i(x, y)\, q - t_i.$$

The total payment $t_X + t_Y$ is the revenue collected by the intermediary. We adopt the following definitions:

Definition 1. An allocation (q, t) consists of a trade rule $q : \mathscr{X} \times \mathscr{Y} \longrightarrow \{0, 1\}$ and a payment rule $t = (t_X, t_Y)$, where $t_i : \mathscr{X} \times \mathscr{Y} \longrightarrow \mathbb{R}$. For any $x \in \mathscr{X}$ and $y \in \mathscr{Y}$, $q(x, y)$ indicates if the transaction takes place or not, and $t_i(x, y)$ is the monetary transfer made by i.

Remark 1. The allocation is deterministic; there is no randomization even if both parties are indifferent between transacting or not.

Definition 2. A mechanism $\langle \mathscr{M}, (Q, T) \rangle$ is a message space $\mathscr{M} = \mathscr{M}_X \times \mathscr{M}_Y$ and a decision rule (Q, T), which consists of a trade decision $Q : \mathscr{M} \longrightarrow \{0, 1\}$ and a payment decision $T = (T_X, T_Y)$, where $T_i : \mathscr{M} \longrightarrow \mathbb{R}$. For any $m = (m_X, m_Y) \in \mathscr{M}$, $Q(m)$ indicates if the transaction takes place or not, and $T_i(m)$ is the monetary transfer made by i.

Combined with $\mathscr{X} \times \mathscr{Y}$ a mechanism $\langle \mathscr{M}, (Q,T) \rangle$ describes a game of incomplete information, in which the intermediary first commits to the mechanism, and then each $x \in \mathscr{X}$ and $y \in \mathscr{Y}$ choose messages $m_X \in \mathscr{M}_X$ and $m_Y \in \mathscr{M}_Y$, respectively. For any pair of messages $m = (m_X, m_Y)$ the decision $(Q(m), T(m))$ determines the payoffs

$$u_i\left(Q(m), T_i(m); x, y\right) = v_i(x,y) Q(m) - T_i(m)$$

and the total payment $T_X(m) + T_Y(m)$. The equilibrium concept is ex post equilibrium in pure strategies $s_X : \mathscr{X} \longrightarrow \mathscr{M}_X$ and $s_Y : \mathscr{Y} \longrightarrow \mathscr{M}_Y$, and $s = (s_X, s_Y)$ denotes a strategy profile.

Definition 3. A strategy profile s^* constitutes an ex post equilibrium of the game if for every $x \in \mathscr{X}$ and $y \in \mathscr{Y}$:

$$s_X^*(x) \in \arg\max_{m_X \in \mathscr{M}_X} u_X\left(Q\left(m_X, s_Y^*(y)\right), T_X\left(m_X, s_Y^*(y)\right); x, y\right),$$
$$s_Y^*(y) \in \arg\max_{m_Y \in \mathscr{M}_Y} u_Y\left(Q\left(s_X^*(x), m_Y\right), T_Y\left(s_X^*(x), m_Y\right); x, y\right).$$

In other words, the strategy profile s^* is an ex post equilibrium if and only if the action profile $(s_X^*(x), s_Y^*(y))$ is a Nash equilibrium for every $(x,y) \in \mathscr{X} \times \mathscr{Y}$. Put differently, ex post equilibrium is a Bayesian equilibrium with no regret: even if the signal observed by one party were to become public, the other party would have no incentive to change action.

We consider both weak and strong ex post implementation:

Definition 4. The allocation (q,t) is *weakly* ex post implementable if there exists a mechanism $\langle \mathscr{M}, (Q,T) \rangle$ for which there exists an ex post equilibrium s^*, such that

$$(q,t) = (Q,T) \circ s^*.$$

Definition 5. The allocation (q,t) is *strongly* ex post implementable if it is weakly ex post implementable and there exists a mechanism $\langle \mathscr{M}, (Q,T) \rangle$, such that for every ex post equilibrium s^* we have

$$(q,t) = (Q,T) \circ s^*.$$

By the Revelation Principle,[2] the allocation (q,t) is weakly ex post implementable if and only if truth-telling constitutes an ex post equilibrium of the direct mechanism $\langle \mathscr{X} \times \mathscr{Y}, (q,t) \rangle$. Lemma 1 characterizes weakly ex post implementable allocations, using the following notation:

$$\underline{v}_Y(x) = \inf_{y \in \mathscr{Y}} v_Y(x,y), \quad \overline{v}_Y(x) = \sup_{y \in \mathscr{Y}} v_Y(x,y),$$

$$\underline{v}_X(y) = \inf_{x \in \mathscr{X}} v_X(x,y), \quad \overline{v}_X(y) = \sup_{x \in \mathscr{X}} v_X(x,y).$$

Lemma 1. *The allocation (q,t) is weakly ex post implementable if and only if there exist:*

- *Price menus $\mathscr{P}_Y = p_Y(\mathscr{X})$ and $\mathscr{P}_X = p_X(\mathscr{Y})$ with $p_Y(x) \in [\underline{v}_Y(x), \overline{v}_Y(x)]$ and $p_X(y) \in [\underline{v}_X(y), \overline{v}_X(y)]$, such that for every $x \in \mathscr{X}$ and $y \in \mathscr{Y}$, the trade rule satisfies:*

$$q(x,y) = 0 \Longrightarrow v_Y(x,y) \le p_Y(x) \text{ and } v_X(x,y) \le p_X(y),$$

$$q(x,y) = 1 \Longrightarrow v_Y(x,y) \ge p_Y(x) \text{ and } v_X(x,y) \ge p_X(y).$$

- *Fee menus $\mathscr{F}_Y = f_Y(\mathscr{X})$ and $\mathscr{F}_X = f_X(\mathscr{Y})$, such that for every $x \in \mathscr{X}$ and*

[2] See Myerson (2013), for instance.

$y \in \mathscr{Y}$, the payments satisfy:

$$t_Y(x,y) = f_Y(x) + p_Y(x)\, q(x,y),$$
$$t_X(x,y) = f_X(y) + p_X(y)\, q(x,y).$$

Proof. We first show that the two-part tariff structure implies that the allocation (q,t) is weakly ex post implementable by the direct mechanism $\langle \mathscr{X} \times \mathscr{Y}, (q,t)\rangle$. For every $(x,y) \in \mathscr{X} \times \mathscr{Y}$ and $m_X \in \mathscr{M}_X = \mathscr{X}$:

$$\begin{aligned} u_X(q(m_X,y), t_X(m_X,y); x,y) &= v_X(x,y)\, q(m_X,y) - t_X(m_X,y) \\ &= [v_X(x,y) - p_X(y)]\, q(m_X,y) - f_X(y) \\ &\le [v_X(x,y) - p_X(y)]\, q(x,y) - f_X(y) \\ &= u_X(q(x,y), t_X(x,y); x,y), \end{aligned}$$

where the inequality follows from the fact that:

$$v_X(x,y) > p_X(y) \Longrightarrow q(x,y) = 1 \ge q(m_X,y),$$
$$v_X(x,y) < p_X(y) \Longrightarrow q(x,y) = 0 \le q(m_X,y).$$

Similarly, for party Y. Thus, telling the truth constitutes an ex post equilibrium of the direct mechanism.

We now show that the two-part tariff structure is necessary for weak ex post implementability. Assume (q,t) is weakly ex post implementable. Then, by the Revelation Principle, truth-telling constitutes an equilibrium of the direct mechanism $\langle \mathscr{X} \times \mathscr{Y}, (q,t)\rangle$.

Consider party X, any $x, \hat{x} \in \mathscr{X}$, and any $y \in \mathscr{Y}$. By ex post incentive compatibility:

$$
\begin{aligned}
v_X(x,y)\, q(x,y) - t_X(x,y) &\geq v_X(x,y)\, q(\hat{x},y) - t_X(\hat{x},y), \\
v_X(\hat{x},y)\, q(\hat{x},y) - t_X(\hat{x},y) &\geq v_X(\hat{x},y)\, q(x,y) - t_X(x,y).
\end{aligned}
$$

Combining the incentive compatibility constraints yields:

$$
\begin{aligned}
v_X(\hat{x},y)\left[q(\hat{x},y) - q(x,y)\right] &\geq t_X(\hat{x},y) - t_X(x,y) \\
&\geq v_X(x,y)\left[q(\hat{x},y) - q(x,y)\right].
\end{aligned}
$$

Thus, $q(\hat{x},y) = q(x,y)$ implies $t_X(\hat{x},y) = t_X(x,y)$. Therefore, the signal x affects $t_X(x,y)$ only through the impact on the trade rule. We can define:

$$
p_X(y) \coloneqq \begin{cases}
\overline{v}_X(y) & \text{if } q(x,y) = 0, \forall x \in \mathscr{X}, \\
\underline{v}_X(y) & \text{if } q(x,y) = 1, \forall x \in \mathscr{X}, \\
t_X(\hat{x},y)\big|_{q(\hat{x},y)=1} - t_X(x,y)\big|_{q(x,y)=0} & \text{otherwise},
\end{cases}
$$

which precisely captures this impact, and

$$
f_X(y) \coloneqq t_X(x,y) - p_X(y)\, q(x,y),
$$

which is indeed independent of x. We now check that $p_X(y) \in [\underline{v}_X(y), \overline{v}_X(y)]$. By construction, this is true when the trade rule is constant over x. If this is not the case, there exist $x, \hat{x} \in \mathscr{X}$, such that $q(x,y) = 0$ and $q(\hat{x},y) = 1$. The incentive compatibility constraints then imply that $v_X(x,y) \leq p_X(y)$ and $v_X(\hat{x},y) \geq p_X(y)$. Thus, indeed, $p_X(y)$ belongs to the interval. Finally, repeating the same steps for the other party, we can use the incentive compatibility constraints of both parties to conclude that

$q(x,y) = 0$ implies $v_X(x,y) \le p_X(y)$ and $v_Y(x,y) \le p_Y(x)$, and $q(x,y) = 1$ implies the reverse of these inequalities. $\square$

Lemma 1 shows that a two-part tariff structure is both necessary and sufficient for weak ex post implementation. The transaction price and the fixed fee paid by one party only depend on the signal observed by the other party; the party's own signal thus affects its payment only through the impact on the trade rule. In other words, each party must simply decide whether or not to trade at the transaction price it faces.

The intuition for Lemma 1 is the following. First, by incentive compatibility, the payments can only depend on the signal observed by the other party and whether or not the transaction occurs. Otherwise the party will have an incentive to send the report that minimizes its payment among those that give the same trading outcome. We can then divide the payments into two parts: a fixed fee paid independently of the transaction and a price paid only if the transaction occurs. One can interpret this argument as an application of the Taxation Principle.[3]

Second, when a given type y always completes the transaction, there is a degree of freedom in choosing the associated two-part tariff. We may choose $p_X(y) = \overline{v}_X(y)$ and adjust the fixed fee $f_X(y)$ accordingly. Likewise, if type y never completes the transaction, we may set $p_X(y) = \underline{v}_X(y)$ as the price is never paid. Incentive compatibility then requires that the transaction prices satisfy the conditions stated in the lemma.

The two-part tariff structure suggests an indirect mechanism, where instead of reporting their type directly, each party quotes a price and a fixed fee, and the intermediary makes a decision based on the prices. Formally, price mechanisms are defined as follows:

Definition 6. A price mechanism $(\mathscr{P}, \mathscr{F}, Q)$ contains price menus $\mathscr{P}$, fee menus $\mathscr{F}$ and a trade decision $Q : \mathscr{P} \longrightarrow \{0,1\}$, such that for any prices $(p_Y, p_X) \in \mathscr{P}$ and

[3]See Salanié (2005) for instance.

fees $(f_Y, f_X) \in \mathscr{F}$, $Q(p_Y, p_X)$ indicates if the transaction takes place or not, and $f_i + p_i Q(p_Y, p_X)$ is the monetary transfer made by i.

In a price mechanism each party chooses a two-part tariff, including the price the other party must pay to compete the transaction and the fixed fee that the other party must pay regardless of the transaction. Given any price quotes, party i has utility

$$u_i\left(Q\left(p_Y, p_X\right), \left(p_i, f_i\right); x, y\right) = \left[v_i\left(x, y\right) - p_i\right] Q\left(p_Y, p_X\right) - f_i.$$

The intermediary's revenue is the sum of the fixed fees, and if trade takes place, it also gets the sum of the transaction prices.

Price mechanisms are particularly simple: both parties quote a simple price instead of reporting their – possibly multi-dimensional – private information. We now turn to the main results of the paper, showing that price mechanisms suffice to capture all the relevant information when the values are negatively interdependent:

Definition 7. The values are negatively interdependent if for every $x, \hat{x} \in \mathscr{X}$ and $y, \hat{y} \in \mathscr{Y}$:

$$v_X\left(\hat{x}, \hat{y}\right) > v_X\left(x, y\right) \Longrightarrow v_Y\left(\hat{x}, \hat{y}\right) < v_Y\left(x, y\right),$$
$$v_Y\left(\hat{x}, \hat{y}\right) > v_Y\left(x, y\right) \Longrightarrow v_X\left(\hat{x}, \hat{y}\right) < v_X\left(x, y\right).$$

Under negative interdependence, if one party observes a higher signal and thus values the transaction more, ceteris paribus, the other party values the transaction less. This property seems realistic in many contexts. In the case of a buyer and a seller, for example, this is equivalent to saying that they have positively correlated signals about the value of the object owned by the seller: a high signal increases the value of the good for both parties, making the seller less willing to trade and the buyer more eager to purchase.

We first analyze ex post implementation in an environment with one-dimensional signals, before turning to more general signal spaces. We then apply the results and conclude.

3 One-dimensional signals

In this section, we suppose that the private information held by each party can be summarized by a single number: $\mathscr{X} = [\underline{x}, \overline{x}]$ and $\mathscr{Y} = [\underline{y}, \overline{y}]$, where $\underline{x} < \overline{x}$ and $\underline{y} < \overline{y}$. We make the following regularity assumption:

Assumption 1. *$v_X(x, y)$ is continuous, strictly increasing in x and $v_Y(x, y)$ is continuous, strictly increasing in y.*

For any trade rule, we define the marginal types:

$$x^q(y) = \inf_{x \in \mathscr{X}} \{q(x, y) = 1\},$$
$$y^q(x) = \inf_{y \in \mathscr{Y}} \{q(x, y) = 1\},$$

adopting the conventions $x^q(y) = \overline{x}$ and $y^q(x) = \overline{y}$ for empty sets. These pin down the transaction price functions, characterized by Lemma 1: $p_X(y) = v_X(x^q(y), y)$ and $p_Y(x) = v_Y(x, y^q(x))$. Thus, by Lemma 1, the marginal types satisfy the cut-off property, that is, all types above the marginal types trade (and those below do not). Moreover, the marginal types are monotonic:

Lemma 2. *The allocation (q, t) is weakly ex post implementable only if the marginal types are weakly decreasing. Furthermore, if the marginal types are continuous, then* *for every $x \in \mathscr{X}$ and $y \in \mathscr{Y}$, they satisfy the inverse relation:*

$$x^q(y) = \begin{cases} \overline{x} & \text{if } y < y^q(\overline{x}), \\ x & \text{if } y = y^q(x), \\ \underline{x} & \text{if } y > y^q(\underline{x}), \end{cases} \qquad y^q(x) = \begin{cases} \overline{y} & \text{if } x < x^q(\overline{y}), \\ y & \text{if } x = x^q(y), \\ \underline{y} & \text{if } x > x^q(\underline{y}). \end{cases}$$

Proof. We first show weak monotonicity. Suppose, by contradiction, that there exists $x, \hat{x} \in \mathscr{X}$, such that $\hat{x} > x$ and $y^q(\hat{x}) > y^q(x)$. Then, there exists $y \in (y^q(x), y^q(\hat{x}))$. By the cut-off property, $y > y^q(x)$ implies $q(x,y) = 1$, which in turn implies $x \geq x^q(y)$. Likewise, $y < y^q(\hat{x})$ implies $q(\hat{x}, y) = 0$ and therefore $\hat{x} \leq x^q(y)$. We then have $x \geq \hat{x}$, which contradicts $\hat{x} > x$. Thus $y^q(\hat{x}) \leq y^q(x)$ must hold.

We now show that continuity of the marginal types implies the inverse relation. For any $y \in \mathscr{Y}$, by weak monotonicity and the cut-off property, $y > y^q(\underline{x})$ implies $x^q(y) = \underline{x}$ and $y < y^q(\overline{x})$ implies $x^q(y) = \overline{x}$. By continuity of x^q, we have $x^q(y^q(\underline{x})) = \underline{x}$ and $x^q(y^q(\overline{x})) = \overline{x}$. Furthermore, by the intermediate value theorem, for any $y \in (y^q(\overline{x}), y^q(\underline{x}))$ there exists $x \in (\underline{x}, \overline{x})$, such that $y = y^q(x)$. Then, for $\epsilon > 0$ infinitesimally small, $y + \epsilon > y^q(x)$ implies $q(x, y+\epsilon) = 1$ and $y - \epsilon < y^q(x)$ implies $q(x, y-\epsilon) = 0$, which in turn imply $x \geq x^q(y+\epsilon)$ and $x \leq x^q(y-\epsilon)$, and thus $x^q(y) = x$ by continuity. $\square$

We use Lemmas 1 and 2 to construct the results for environments with negative interdependence, private values and positive interdependence in this sequence.

3.1 Negative interdependence

Under Assumption 1, negative interdependence means that $v_X(x,y)$ is strictly decreasing in y and $v_Y(x,y)$ is strictly decreasing in x. Together with the two-part tariff structure, negative interdependence implies strictly decreasing transaction price functions:

Lemma 3. *For negatively interdependent values the transaction price functions characterized by Lemma 1 are strictly decreasing.*

Proof. For any $x, \hat{x} \in \mathscr{X}$ such that $\hat{x} > x$ Lemma 2 implies $y^q(\hat{x}) \leq y^q(x)$. By negative interdependence and Assumption 1 $\hat{x} > x$ and $y^q(\hat{x}) \leq y^q(x)$ then imply $v_Y(\hat{x}, y^q(\hat{x})) < v_Y(x, y^q(x))$. Therefore $p_Y(\hat{x}) < p_Y(x)$. $\square$

Lemma 3 uses monotonicity of the marginal types to show that the transaction price functions must be strictly decreasing for the allocation to be weakly ex post implementable under negative interdependence. Intuitively, if a low type completes the transaction, incentive compatibility requires that all higher types will also trade, because they value the transaction more and would otherwise mimic the low type. However, when the values are negatively interdependent, the other party benefits less from a transaction with a higher type, implying that the price must decrease to conform with incentive compatibility.

As prices are strictly decreasing, for every price paid by one party, there exists a unique signal observed by the other party. Such injectivity yields the result on weak ex post implementability:

Proposition 1. *Under negative interdependence the allocation is weakly ex post implementable if and only if there exists a price mechanism that weakly ex post implements it.*

Proof. Suppose the values are negatively interdependent and that the allocation (q, t) is weakly ex post implementable. By the Revelation Principle, (q, t) is weakly implementable by the direct mechanism $\langle \mathscr{X} \times \mathscr{Y}, (q, t) \rangle$, truth-telling being an ex post equilibrium. By Lemma 1, a satisfies the two-part tariff structure, and by Lemma 3,

$p_X(y)$ is strictly decreasing in y and $p_Y(x)$ is strictly decreasing in x. Thus, there exist one-to-one mappings $\gamma : \mathscr{P}_X \longrightarrow \mathscr{Y}$ and $\chi : \mathscr{P}_Y \longrightarrow \mathscr{X}$, such that $\gamma(p_X(y)) = y$ for each $y \in \mathscr{Y}$ and $\chi(p_Y(x)) = x$ for each $x \in \mathscr{X}$. Furthermore, there exists a price mechanism $(\mathscr{P}, \mathscr{F}, Q)$ with $\mathscr{P} = \mathscr{P}_Y \times \mathscr{P}_X$ and $\mathscr{F} = \mathscr{F}_Y \times \mathscr{F}_X$, such that:

$$Q(p_Y, p_X) = q(\chi(p_Y), \gamma(p_X)) \text{ for each } (p_Y, p_X) \in \mathscr{P}.$$

As truth-telling is an ex post equilibrium of the direct mechanism, reporting the associated prices and fixed fees constitutes an ex post equilibrium of the price mechanism. For each $x \in \mathscr{X}$ and $y \in \mathscr{Y}$:

$$Q(p_Y(x), p_X(y)) = q(\chi(p_Y(x)), \gamma(p_X(y))) = q(x, y).$$

We may thus conclude that if the allocation is weakly ex post implementable, then there exists a price mechanism that weakly ex post implements it. The reverse is trivial: if there exists a price mechanism that weakly ex post implements the allocation, then the allocation is weakly ex post implementable. □

Remark 2. Strict monotonicity of the price functions implies that the intermediary obtains lower prices (and hence a lower total price) from parties who observe high signals about the value of the transaction. In equilibrium, a party with a high signal sets a lower price to the other party. This highlights the cost of eliciting private information from the parties and the fundamental conflict between efficiency and profit-maximization: the intermediary obtains more revenue from transactions that generate less value.

3.2 Private values

Proposition 1 does not apply to the environment with private values: $v_X(x,y) = x$ and $v_Y(x,y) = y$. In this case, the transaction price functions characterized by Lemma 1 equal the marginal types: $p_X = x^q$ and $p_Y = y^q$. They are thus weakly decreasing by Lemma 2 and can contain flat parts. Hence, different types associated with the same transaction price, may face different decisions about the transaction. As for private values weak ex post implementability is equivalent to implementation in weakly dominant strategies, we have:

Proposition 2. *With private values, there exists an allocation that is implementable in weakly dominant strategies, but cannot be weakly dominant strategy implemented by any price mechanism.*

Proof. Define:

$$\hat{x} = \frac{\underline{x} + \overline{x}}{2}, \quad \hat{y} = \frac{3\underline{y} + \overline{y}}{4}, \quad \tilde{y} = \frac{\underline{y} + 3\overline{y}}{4} \text{ and } b = \frac{\overline{y} - \underline{y}}{\overline{x} - \underline{x}}.$$

Consider an allocation (q,t), such that for every $y \in \mathscr{Y}$ and $x \in \mathscr{X}$, $q(x,y) = 1$ if and only if $y > y^q(x)$, where

$$y^q(x) = \begin{cases} \overline{y} - \frac{b}{2}(x - \underline{x}) & \text{if } x < \hat{x}, \\ \frac{\underline{y}+\overline{y}}{2} & \text{if } x = \hat{x}, \\ \underline{y} + \frac{b}{2}(\overline{x} - x) & \text{if } x > \hat{x}. \end{cases}$$

Furthermore, define $t_Y(x,y) = y^q(x)\,q(x,y)$ and $t_X(x,y) = x^q(y)\,q(x,y)$ with

$$x^q(y) = \begin{cases} \overline{x} - \frac{2}{b}\left(y - \underline{y}\right) & \text{if } y < \hat{y}, \\ \hat{x} & \text{if } y \in [\hat{y}, \tilde{y}], \\ \underline{x} + \frac{2}{b}\left(\overline{y} - y\right) & \text{if } y > \tilde{y}. \end{cases}$$

Note that the allocation satisfies Lemma 1, and is therefore weakly implementable in dominant strategies. However, $p_X(\hat{y}) = p_X(\tilde{y})$, although $q(\hat{x}, \tilde{y}) = 1$ and $q(\hat{x}, \hat{y}) = 0$. No price mechanism can implement such allocation, as this would require two types to be associated with the same equilibrium message, and yet have different trades. □

As private values is a borderline case, this suggests that with private values, price mechanisms should implement almost every weakly dominant strategy implementable allocation. Indeed, a flat part in one of the price functions is necessarily associated with a discontinuity point in the other. As the transaction prices are given by the marginal types, by focusing on allocations with continuous marginal types we obtain continuous price functions that satisfy the inverse relation in Lemma 2. Using this property, we obtain the following result:

Proposition 3. *Suppose the marginal types are continuous. Then, under private values, the allocation is weakly dominant strategy implementable if and only if there exists a price mechanism that implements it in weakly dominant strategies.*

Proof. See the Appendix. □

The proof of Proposition 3 uses the fact that if the transaction price is constant over a range of types, then so is the trade rule.

Restricting attention to allocations with continuous marginal types appears to be quite innocuous. Indeed, as long as the probability density function over the types is continuous, optimal allocations have continuous marginal types. In particular, efficient allocations satisfy this property:

Example 1. Consider ex post efficient allocations; that is, allocations with a trade rule, such that $x+y>0$ implies $q(x,y)=1$ and $x+y<0$ implies $q(x,y)=0$. The marginal types, and hence the transaction price functions, are continuous:

$$p_X(y)=\begin{cases}\overline{x} & \text{if } -y>\overline{x},\\ -y & \text{if } \underline{x}\leq -y\leq \overline{x},\\ \underline{x} & \text{if } -y<\underline{x},\end{cases} \qquad p_Y(x)=\begin{cases}\overline{y} & \text{if } -x>\overline{y},\\ -x & \text{if } \underline{y}\leq -x\leq \overline{y},\\ \underline{y} & \text{if } -x<\underline{y}.\end{cases}$$

To construct the price mechanism, choose price menus $\mathscr{P}_Y=-\mathscr{X}$ and $\mathscr{P}_X=-\mathscr{Y}$, and a trade decision, such that for every $p_Y\in-\mathscr{X}$ and $p_X\in-\mathscr{Y}$ we have:

$$Q(p_Y,p_X)=q(-p_Y,-p_X)=\begin{cases}1 & \text{if } p_Y+p_X<0,\\ 0 & \text{if } p_Y+p_X>0.\end{cases}$$

Furthermore, select the following fee menus:

$$\mathscr{F}_Y=\left\{f_Y(x)+\max\left\{x+\underline{y},0\right\}:x\in\mathscr{X}\right\},$$
$$\mathscr{F}_X=\left\{f_X(y)+\max\left\{y+\underline{x},0\right\}:y\in\mathscr{Y}\right\}.$$

The proof of Proposition 3 uses similar construction.

3.3 Positive interdependence

The implementation result does not extend to positive interdependence, however. Even with continuous marginal types, price mechanisms then fail to implement weakly ex post implementable allocations:

Proposition 4. *Suppose that* $v_X(x,y) = x + \epsilon y$*, where* $\epsilon > 0$*, however small, and* $v_Y(x,y) = y$*. Then, there exists a weakly ex post implementable allocation that has continuous marginal types and cannot be weakly ex post implemented by any price mechanism.*

Proof. Consider an allocation (q,t), such that for every $y \in \mathscr{Y}$ and $x \in \mathscr{X}$, $q(x,y) = 1$ if and only if $y > y^q(x)$, where

$$y^q(x) = \begin{cases} \overline{y} & \text{if } x \leq \overline{x} - \epsilon\left(\overline{y} - \underline{y}\right), \\ \underline{y} + (\overline{x} - x)/\epsilon & \text{if } x > \overline{x} - \epsilon\left(\overline{y} - \underline{y}\right). \end{cases}$$

Define $t_Y(x,y) = v_Y(x, y^q(x))\, q(x,y)$ and $t_X(x,y) = v_X(x^q(y), y)\, q(x,y)$ with $x^q(y) = \overline{x} - \epsilon\left(y - \underline{y}\right)$. By Lemma 1, the allocation is weakly ex post implementable. Yet, for any $x > \overline{x} - \epsilon\left(\overline{y} - \underline{y}\right)$ we have:

$$\begin{aligned} p_Y(x) &= v_Y(x, y^q(x)) \\ &= x + \epsilon y^q(x) \\ &= x + \epsilon\left[\underline{y} + (\overline{x} - x)/\epsilon\right] \\ &= \epsilon\underline{y} + \overline{x}, \end{aligned}$$

which is a constant. As $q(x,y)$ depends on x, the constructed allocation cannot be implemented by any price mechanism. $\square$

4 Multi-dimensional signals

Let us now relax the assumption of one-dimensionality and consider more general signal spaces. We will however impose a separability assumption:

Assumption 2. *For any $x \in \mathscr{X}$ and $y \in \mathscr{Y}$, the values can be written as:*

$$v_X(x,y) = \phi_X(x) + \psi_X(y),$$
$$v_Y(x,y) = \phi_Y(x) + \psi_Y(y),$$

where $\phi_i : \mathscr{X} \longrightarrow \mathbb{R}$ and $\psi_i : \mathscr{Y} \longrightarrow \mathbb{R}$.

Without this assumption, if both parties have multi-dimensional signals, the generic impossibility result of Jehiel et al. (2006) obtains.

4.1 Weak ex post implementation

For the purpose of extending Proposition 1, suppose that the signal spaces are connected and that the value functions are continuous. Furthermore, we consider the following class of allocations:

Definition 8. The allocation (q,t) is value-driven if for any $x, \hat{x} \in \mathscr{X}$ and $y, \hat{y} \in \mathscr{Y}$:

$$\left.\begin{array}{l} v_X(x,y) = v_X(\hat{x},\hat{y}) \\ v_Y(x,y) = v_Y(\hat{x},\hat{y}) \end{array}\right\} \Longrightarrow q(x,y) = q(\hat{x},\hat{y}).$$

This holds trivially in the one-dimensional environment with additively separable values, because the transaction values can be the same only if the signals are the same.

We have the following lemma:

Lemma 4. *Under negatively interdependent values the allocation (q, t) is weakly ex post implementable only if for every $x, \hat{x} \in \mathscr{X}$ and $y, \hat{y} \in \mathscr{Y}$:*

$$\left. \begin{array}{l} p_X(y) = p_X(\hat{y}) \\ p_Y(x) = p_Y(\hat{x}) \end{array} \right\} \Longrightarrow \left. \begin{array}{l} v_X(x, y) = v_X(\hat{x}, \hat{y}) \\ v_Y(x, y) = v_Y(\hat{x}, \hat{y}) \end{array} \right\}.$$

Proof. See the Appendix. □

With this lemma we can prove the following result:

Proposition 5. *Under negatively interdependent values, any value-driven allocation is weakly ex post implementable if and only if there exists a price mechanism that weakly ex post implements it.*

Proof. Suppose the allocation (q, t) is weakly ex post implementable and value-driven. Then, by Lemma 1, the allocation satisfies the two-part tariff structure. Define $\mathscr{P} = \mathscr{P}_Y \times \mathscr{P}_X$ as stated in Lemma 1. As the allocation is value-driven, by Lemma 4 the trade rule q depends on x and y only through $p_Y(x)$ and $p_X(y)$. Thus, there exists a trade rule $Q : \mathscr{P} \longrightarrow \{0, 1\}$, such that for every $x \in \mathscr{X}$ and $y \in \mathscr{Y}$

$$Q(p_Y(x), p_X(y)) = q(x, y).$$

As truth-telling is an ex post equilibrium of the direct mechanism, reporting the associated transaction prices constitutes an ex post equilibrium of the price mechanism $(\mathscr{P}, \mathscr{F}, Q)$. Hence, if the allocation is weakly ex post implementable, then there exists a price mechanism that weakly ex post implements it. □

Proposition 5 extends Proposition 1 to multi-dimensional signals when values are negatively interdependent. For the case of private values the extension is essentially a matter of relabeling. We have $v_X(x, y) = \phi_X(x)$ and $v_Y(x, y) = \psi_Y(y)$, and as we focus on value-driven allocations, we can define marginal types with respect to the values and obtain:

Proposition 6. *Suppose the marginal types are continuous. Then, under private values, any value-driven allocation is weakly ex post implementable if and only if there exists a price mechanism that weakly ex post implements it.*

Proof. Relabel $x = \phi_X$ and $y = \psi_Y$ to apply Proposition 3 directly. □

4.2 Strong ex post implementation

Let us now consider strong ex post implementation. Together with Assumption 2 and negatively interdependent values, strong ex post implementation implies injectivity:

Lemma 5. *Under negative interdependence the allocation (q, t) is strongly ex post implementable only if for any $x, \hat{x} \in \mathscr{X}$ and $y, \hat{y} \in \mathscr{Y}$:*

$$p_X(\hat{y}) = p_X(y) \Longrightarrow q(x, \hat{y}) = q(x, y) \text{ and } f_X(\hat{y}) = f_X(y),$$
$$p_Y(\hat{x}) = p_Y(x) \Longrightarrow q(\hat{x}, y) = q(x, y) \text{ and } f_Y(\hat{x}) = f_Y(x).$$

Proof. See the Appendix. □

The idea for the proof of Lemma 5 is the following. If two types associated with the same transaction price have different decisions about the transaction for some type of the other party, then under negative interdependence incentive compatibility implies that both types must be indifferent between completing that transaction or not. But then, strong ex post implementability requires that the other party will deviate from the equilibrium if the types mimic each other, without upsetting the truth-telling equilibrium. This then contradicts the separability assumption.

Lemma 5 directly gives us the following result on equivalence between direct implementation and implementation by price mechanisms:

Proposition 7. *Under negative interdependence the allocation is strongly ex post implementable by the direct mechanism if and only if it is strongly ex post implementable by a price mechanism.*

Proof. For any strongly ex post implementable allocation (q, t), by Lemma 5 there exists a price mechanism $(\mathscr{P}, \mathscr{F}, Q)$ with price menus $\mathscr{P} = \mathscr{P}_Y \times \mathscr{P}_X$ as stated in Lemma 1, such that for every $x \in \mathscr{X}$ and $y \in \mathscr{Y}$ we have $Q\left(p_Y\left(x\right), p_X\left(y\right)\right) = q\left(x, y\right)$. Thus, the allocation being strongly ex post implementable by the direct mechanism $\langle \mathscr{X} \times \mathscr{Y}, (q, t) \rangle$ is equivalent to it being strongly ex post implementable by the price mechanism. $\square$

Proposition 7 permits the existence of an allocation, which is strongly ex post implementable by an indirect mechanism, but not necessarily strongly ex post implementable by the direct mechanism nor any price mechanism. Still, the equivalence result is useful in showing that in environments with multi-dimensional signals, we can replace direct revelation with simple price mechanisms.

5 Applications

5.1 Negative result

So far, we have assumed no constraints other than ex post incentive compatibility. However, given the equilibrium concept, it seems natural to further restrict the set of allocations by imposing the following two ex post constraints:

Definition 9. The allocation (q, t) is ex post individually rational if for each $i = X, Y$

and every $x \in \mathscr{X}$ and $y \in \mathscr{Y}$:

$$v_i(x, y)\, q(x, y) \geq t_i(x, y).$$

Definition 10. The allocation (q, t) is ex post budget balanced if for every $x \in \mathscr{X}$ and $y \in \mathscr{Y}$ the total payment is zero:

$$t_X(x, y) + t_Y(x, y) = 0.$$

Ex post individual rationality requires that, in every state of information, the parties are free to walk away and exercise their outside option, which we normalize to zero. Ex post budget balance instead requires that no outside sink or source of funds is needed.

The following result shows that under negative interdependence, the set of ex post incentive compatible, individually rational and budget balanced allocations is essentially empty:

Proposition 8. *Suppose the values are negatively interdependent. Then, the allocation is weakly ex post implementable, ex post individually rational and budget balanced if and only if one of the following three conditions hold:*

1. *The transaction values are always positive, the transaction is always completed and the payments are zero;*

2. *The transaction is never completed and the payments are zero;*

3. *Only the highest types complete the transaction and pay their valuations, and these valuations sum up to zero.*

Proof. See the Appendix. □

Proposition 8 should be contrasted with the result obtained by Hagerty and Rogerson (1987). They show that under private values, posted price mechanisms are essentially the only mechanisms that satisfy dominant strategy incentive compatibility, ex post individual rationality and budget balance. In a posted price mechanism, the intermediary sets a fixed transaction price and if both parties agree to trade, the buyer pays that price to the seller. Indeed, as Lemma 2 implies, with private values the transaction price functions are weakly decreasing, suggesting that they must be constant and sum to zero to achieve ex post budget balance.

However, as Lemma 3 shows, under negative interdependence the transaction price functions have to be strictly decreasing. This provides the intuition for Proposition 8, which shows that only very particular allocations satisfy all three constraints under negative interdependence. Hence, if the buyer and the seller have positively correlated signals about the value of the good, and thus negatively interdependent values for the transaction, there is no way to design a meaningful trading institution that satisfies all three constraints.

5.2 Bayesian implementation under private values

Assuming linear utilities and private values, Gershkov et al. (2013) show that for any Bayesian incentive compatible mechanism, there exists a dominant strategy incentive compatible mechanism, which is equivalent in the sense that it gives the same expected revenue and the same interim expected payoffs for all agents. Together with Proposition 3 this suggests that we should be able to construct price mechanisms that are optimal in the wider class of Bayesian incentive compatible mechanisms.

Suppose that the intermediary seeks to maximize a weighted sum of the parties' surpluses and revenue from intermediation, given a prior over the type spaces. Following Myerson and Satterthwaite (1983), consider the one-dimensional private-value environment and assume that the valuations are independently distributed according to cumulative distributions G_X and G_Y with positive and continuous densities g_X and g_Y on their supports. Using the terminology in Loertscher and Marx (2019), define the weighted virtual valuations:

$$\gamma_X^{\alpha}(x) = x - (1-\alpha)\frac{1-G_X(x)}{g_X(x)},$$
$$\gamma_Y^{\beta}(y) = y - (1-\beta)\frac{1-G_Y(y)}{g_Y(y)},$$

where $\alpha, \beta \in [0,1]$ denote the bargaining weights of the parties. We assume that these virtual valuations are strictly increasing and adopt the following definitions:

$$\bar{u}_X(x) = x\int_{\underline{y}}^{\overline{y}} q(x,y)\, g_Y(y)\,\mathrm{d}y - \int_{\underline{y}}^{\overline{y}} t_X(x,y)\, g_Y(y)\,\mathrm{d}y,$$
$$\bar{u}_Y(y) = y\int_{\underline{x}}^{\overline{x}} q(x,y)\, g_X(x)\,\mathrm{d}x - \int_{\underline{x}}^{\overline{x}} t_Y(x,y)\, g_X(x)\,\mathrm{d}x,$$

Definition 11. The allocation (q,t) is Bayesian incentive compatible if for every $x, \hat{x} \in \mathscr{X}$ and $y, \hat{y} \in \mathscr{Y}$:

$$\bar{u}_X(x) \geq x\int_{\underline{y}}^{\overline{y}} q(\hat{x},y)\, g_Y(y)\,\mathrm{d}y - \int_{\underline{y}}^{\overline{y}} t_X(\hat{x},y)\, g_Y(y)\,\mathrm{d}y,$$
$$\bar{u}_Y(y) \geq y\int_{\underline{x}}^{\overline{x}} q(x,\hat{y})\, g_X(x)\,\mathrm{d}x - \int_{\underline{x}}^{\overline{x}} t_Y(x,\hat{y})\, g_X(x)\,\mathrm{d}x.$$

Definition 12. The allocation (q,t) is ex interim individually rational if for every $x \in \mathscr{X}$ and $y \in \mathscr{Y}$:

$$\bar{u}_X(x) \geq 0,$$
$$\bar{u}_Y(y) \geq 0.$$

Subject to Bayesian incentive compatibility and ex interim individual rationality, the intermediary seeks to maximize the weighted welfare:

$$\begin{aligned}W^{\alpha,\beta} = &\int_{\underline{y}}^{\overline{y}} \int_{\underline{x}}^{\overline{x}} \left[t_X(x,y) + t_Y(x,y)\right] g_X(x)\, g_Y(y)\, \mathrm{d}x\mathrm{d}y \\ &+ \alpha \int_{\underline{x}}^{\overline{x}} \bar{u}_X(x)\, g_X(x)\, \mathrm{d}x + \beta \int_{\underline{y}}^{\overline{y}} \bar{u}_Y(y)\, g_Y(y)\, \mathrm{d}y.\end{aligned}$$

If the weights are zero, then the intermediary maximizes its own revenue in the usual sense. Full weights amount to maximizing total welfare.

Applying Gershkov et al. (2013), we may replace the Bayesian incentive compatibility constraint with dominant strategy incentive compatibility. By Lemma 1, the optimal allocation must then satisfy the two-part tariff structure and after standard computations, we obtain

$$W^{\alpha,\beta} = \int_{\underline{x}}^{\overline{x}} \int_{p_Y(x)}^{\overline{y}} \left[\gamma_X^{\alpha}(x) + \gamma_Y^{\beta}(y)\right] g_Y(y)\, g_X(x)\, \mathrm{d}y\mathrm{d}x. \tag{1}$$

Thus, fixing x, trade always takes place if $\gamma_X^{\alpha}(x) > -\gamma_Y^{\beta}(\overline{y})$ and it never occurs if $\gamma_X^{\alpha}(x) < -\gamma_Y^{\beta}(\underline{y})$. Otherwise, the optimal transaction price $p_Y^m(x)$ satisfies the weighted one-sided monopoly pricing rule

$$p_Y^m(x) = -\gamma_X^{\alpha}(x) + (1-\beta)\frac{1 - G_Y(p_Y^m(x))}{g_Y(p_Y^m(x))}, \tag{2}$$

being thus strictly decreasing in x. The other price $p_X^m(y)$ is given by the inverse. We have the following result:

Proposition 9. *Suppose the values are private and that the weighted virtual valuations are strictly decreasing. Then, there exists a dominant strategy incentive compatible price mechanism, which maximizes the weighted expected welfare among Bayesian incentive* *compatible and ex interim individually rational mechanisms.*

Proof. See the Appendix. $\square$

Example 2. In the canonical example of bilateral trade, the seller's cost $-x$ and the buyer's valuation y are uniformly and independently distributed along the unit interval. The revenue-maximizing transaction prices are then:

$$p_Y^m(x) = \begin{cases} -x + \frac{1}{2} & \text{if } -x \leq \frac{1}{2}, \\ 1 & \text{if } -x > \frac{1}{2}, \end{cases}$$

and

$$-p_X^m(y) = \begin{cases} 0 & \text{if } y < \frac{1}{2}, \\ y - \frac{1}{2} & \text{if } y \geq \frac{1}{2}, \end{cases}$$

where $p_Y^m(x)$ gives the price charged to the buyer as a function of the seller's cost and $-p_X^m(y)$ is the price paid to the seller as a function of the buyer's valuation. The associated price mechanism has price lists $\mathscr{P}_Y = \left[\frac{1}{2}, \frac{3}{2}\right]$ for the seller and $\mathscr{P}_X = \left[-\frac{1}{2}, \frac{1}{2}\right]$ for the buyer. If trade occurs, the buyer pays p_Y (what the seller asks) and the seller receives $-p_X$ (what the buyer offers). Trade occurs if and only if the buyer's offer is not too much below the seller's ask price: $p_Y - (-p_X) < \frac{1}{2}$ implies trade and $p_Y - (-p_X) > \frac{1}{2}$ no trade. The difference $p_Y - (-p_X)$ is always positive by definition, and it goes to the intermediary. Indeed, it is a dominant strategy for the seller to ask $p_Y^m(x)$ and for the buyer to offer $-p_X^m(y)$, maximizing the intermediary's expected revenue in the class of Bayesian incentive compatible and ex interim individually rational mechanisms.

5.3 Tax incidence

Recently, Weyl and Fabinger (2013) extended the principles of tax incidence under perfect competition to models of imperfect competition. However, their analysis relies on complete information. The purpose of this section is to revisit the principles in the context of ex post incentive compatible trading mechanisms, using the results on implementation through price mechanisms.

The most basic principle of tax incidence, due to Jenkin (1872), states that the economic incidence of the tax does not depend on the identity of the taxpayer. To extend this principle to the current framework, suppose that each party $i = X, Y$ must pay a tax τ_i if the transaction takes place. Furthermore, suppose that a per transaction tax τ_I is directly levied on the intermediary. We have:

Proposition 10. *Fix a total transaction tax* $\tau_X + \tau_Y + \tau_I$ *and any weakly ex post implementable allocation* (q, t)*. Then, for any* $(\tilde{\tau}_X, \tilde{\tau}_Y, \tilde{\tau}_I)$*, such that* $\tilde{\tau}_X + \tilde{\tau}_Y + \tilde{\tau}_I = \tau_X + \tau_Y + \tau_I$*, there exists an allocation* $\left(q, \tilde{t}\right)$ *that is weakly ex post implementable and gives the same ex post payoffs.*

Proof. See the Appendix. □

Thus, without loss of generality, we can follow the convention that the intermediary directly bears the tax and passes it on to the parties through changes in the equilibrium transaction prices.

In what follows, we focus on the one-dimensional environment with weakly negatively interdependent values. Furthermore, we suppose that the types are independently distributed according to distributions G_X and G_Y with continuous densities g_X and g_Y as in the previous section. Assuming that the valuations are continuously differentiable, we define the weighted virtual valuations:

$$V_X^{\alpha}(x,y) = v_X(x,y) - (1-\alpha)\frac{\partial v_X(x,y)}{\partial x}\frac{1-G_X(x)}{g_X(x)},$$
$$V_Y^{\beta}(x,y) = v_Y(x,y) - (1-\beta)\frac{\partial v_Y(x,y)}{\partial y}\frac{1-G_Y(y)}{g_Y(y)},$$

where $\alpha, \beta \in [0,1]$ measure the bargaining weights of the parties as above. Finally, we assume that the total virtual valuation, which we denote by $V^{\alpha,\beta}(x,y)$ to shorten notation, is strictly increasing in both arguments.

Subject to ex post incentive compatibility and ex interim individual rationality of the parties, we suppose that the intermediary maximizes the weighted welfare

$$W^{\alpha,\beta}(q,t;\tau) = R(t;\tau) + \alpha U_X(q,t) + \beta U_Y(q,t),$$

where $R(t;\tau)$ denotes the expected revenue of the intermediary and $U_i(q,t)$ the expected utility of party i, as a function of the allocation:

$$\begin{aligned} U_i(q,t) &= \int_{\underline{y}}^{\overline{y}} \int_{\underline{x}}^{\overline{x}} \left[v_i(x,y)\, q(x,y) - t_i(x,y)\right] g_X(x)\, g_Y(y)\, \mathrm{d}x\mathrm{d}y, \\ R(t;\tau) &= \int_{\underline{y}}^{\overline{y}} \int_{\underline{x}}^{\overline{x}} \left[t_X(x,y) + t_Y(x,y) - \tau\right] g_X(x)\, g_Y(y)\, \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Note that the per transaction tax, $\tau \geq 0$, enters directly only to the revenue of the intermediary and will affect the utilities of the parties indirectly through the optimal allocation.

By Lemma 1, the optimal allocation, which we denote by $(q(\cdot;\tau), t(\cdot;\tau))$, satisfies the two-part tariff structure. In particular, the interim expected utility of type $\underline{x}$ satisfies

$$0 \geq \int_{y^q(\underline{x};\tau)}^{\overline{y}} \left[v_X(\underline{x},y) - p_X(y;\tau)\right] g_Y(y)\, \mathrm{d}y \geq \int_{\underline{y}}^{\overline{y}} f_X(y;\tau)\, g_Y(y)\, \mathrm{d}y,$$

where the first inequality follows from $p_X(y;\tau) \geq v_X(\underline{x},y)$, by Lemma 1, and the second one from individual rationality. By optimality, the expected fixed fee is then zero, and without loss of generality, we set $f_X(\cdot;\tau) = 0$. Similarly, $f_Y(\cdot;\tau) = 0$.

To express the weighted welfare in terms of the total virtual valuation, note that:

$$\begin{aligned} U_Y(q,t) &= \int_{\underline{y}}^{\overline{y}} \int_{x^q(y;\tau)}^{\overline{x}} \left[v_Y(x,y) - p_Y(x;\tau)\right] g_X(x)\, g_Y(y)\, \mathrm{d}x\mathrm{d}y \\ &= \int_{\underline{x}}^{\overline{x}} \int_{y^q(x;\tau)}^{\overline{y}} \left[v_Y(x,y) - p_Y(x;\tau)\right] g_Y(y)\, g_X(x)\, \mathrm{d}y\mathrm{d}x, \end{aligned}$$

where the second equality follows from Lemma 1 and Fubini's theorem, allowing us to switch the order of integration. Using integration by parts:

$$U_Y(q,t) = \int_{\underline{x}}^{\overline{x}} \int_{y^q(x;\tau)}^{\overline{y}} \frac{\partial v_Y(x,y)}{\partial y} \left[1 - G_Y(y)\right] g_X(x)\, \mathrm{d}y\mathrm{d}x. \tag{3}$$

Repeating the same steps for the other party and using the fact that the expected revenue can be written as total welfare less the utilities of the parties, we obtain

$$W^{\alpha,\beta}(q,t;\tau) = \int_{\underline{y}}^{\overline{y}} \int_{x^q(y;\tau)}^{\overline{x}} \left[V^{\alpha,\beta}(x,y) - \tau\right] g_X(x)\, g_Y(y)\, \mathrm{d}x\mathrm{d}y. \tag{4}$$

Thus, fixing x, trade always takes place if $V^{\alpha,\beta}\left(x,\underline{y}\right) > \tau$; conversely, trade never takes place if $V^{\alpha,\beta}(x,\overline{y}) < \tau$. In the latter case a tax increase has no effect, whereas in the former case the pass-through rates are equal to one.

In what follows we focus on the remaining cases, in which trade occurs if y exceeds the marginal type $y^q(x;\tau)$ that optimally satisfies $V^{\alpha,\beta}(x, y^q(x;\tau)) = \tau$. This pins down the optimal transaction price: $p_Y(x;\tau) = v_Y(x, y^q(x;\tau))$. For any price p_Y and any given x, define the marginal type $\hat{y}(p_Y;x)$ by the identity $p_Y = v_Y(x, \hat{y}(p_Y;x))$ and the associated demand by

$$D_Y(p_Y;x) \coloneqq 1 - G_Y(\hat{y}(p_Y;x)).$$

In equilibrium, $\hat{y}\left(p_Y\left(x;\tau\right);x\right)=y^q\left(x;\tau\right)$, which leads to

$$D_Y\left(p_Y\left(x;\tau\right);x\right)=1-G_Y\left(y^q\left(x;\tau\right)\right). \tag{5}$$

The demand $D_X\left(p_X;y\right)$ is defined analogously with similar identities. Expected amount of trade can then be calculated by taking the expectation of either of the demands:

$$D\left(t\right)=\int_{\underline{y}}^{\overline{y}} D_X\left(p_X\left(y;\tau\right);y\right)g_Y\left(y\right)\mathrm{d}y=\int_{\underline{x}}^{\overline{x}} D_Y\left(p_Y\left(x;\tau\right);x\right)g_X\left(x\right)\mathrm{d}x.$$

Finally, we will denote the pass-through rates by

$$\rho_X\left(y;\tau\right)=\frac{\partial p_X\left(y;\tau\right)}{\partial\tau}\text{ and }\rho_Y\left(x;\tau\right)=\frac{\partial p_Y\left(x;\tau\right)}{\partial\tau}.$$

As the total virtual valuation is strictly increasing, the optimality condition implies that $y^q\left(x;\tau\right)$, and therefore $p_Y\left(x;\tau\right)$, increases with τ. Hence, the parties are going to face a higher transaction price due to a tax increase:

Proposition 11. *A small increase in the per transaction tax changes the expected utilities of the parties by:*

$$\begin{aligned}\frac{\partial U_X}{\partial\tau}&=-\int_{\underline{y}}^{\overline{y}}\rho_X\left(y\right)D_X\left(p_X\left(y;\tau\right);y\right)g_Y\left(y\right)\mathrm{d}y,\\ \frac{\partial U_Y}{\partial\tau}&=-\int_{\underline{x}}^{\overline{x}}\rho_Y\left(x\right)D_Y\left(p_Y\left(x;\tau\right);x\right)g_X\left(x\right)\mathrm{d}x,\end{aligned}$$

and the weighted welfare by

$$\frac{\partial W^{\alpha,\beta}}{\partial\tau}=-D\left(t\right).$$

The intermediary's expected revenue is changed by:

$$\begin{aligned}\frac{\partial R}{\partial \tau} &= -D(t) \\ &\quad + \alpha \int_{\underline{y}}^{\overline{y}} \rho_X(y)\, D_X(p_X(y;\tau);y)\, g_Y(y)\, \mathrm{d}y \\ &\quad + \beta \int_{\underline{x}}^{\overline{x}} \rho_Y(x)\, D_Y(p_Y(x;\tau);x)\, g_X(x)\, \mathrm{d}x.\end{aligned}$$

Proof. From (3) we obtain:

$$\begin{aligned}\frac{\partial U_Y}{\partial \tau} &= -\int_{\underline{x}}^{\overline{x}} \frac{\partial v_Y(x, y^q(x;\tau))}{\partial y} \left[1 - G_Y(y^q(x;\tau))\right] \frac{\partial y^q(x;\tau)}{\partial \tau} g_X(x)\, \mathrm{d}x \\ &= -\int_{\underline{x}}^{\overline{x}} \frac{\partial p_Y(x;\tau)}{\partial \tau} D_Y(p_Y(x;\tau);x)\, g_X(x)\, \mathrm{d}x,\end{aligned}$$

where the second equality follows from (5) and from $p_Y(x;\tau) = v_Y(x, y^q(x;\tau))$ by taking the derivative with respect to τ. We obtain the effect to X by an analogous argument. Furthermore, as the intermediary maximizes (4) without any restrictions, we can apply the envelope theorem to conclude that the effect on the objective is $-D(t)$. Finally, the impact on expected revenue follows directly from the definition of weighted welfare. $\square$

We then directly obtain the second principle of tax incidence; that is, that the total burden of the infinitesimal tax is shared between the parties and the intermediary. Importantly, the impact of the tax on weighted welfare is always equal to the mechanical impact of the tax. Thus, when the intermediary maximizes the total surplus, there is no excess burden of the tax. By contrast, when the intermediary maximizes its own revenue, it bears exactly the mechanical impact and the entire impact of the tax to the parties is additional.

This brings us to the third principle. The incidence of the infinitesimal tax, that is, the ratio of the change in the total surplus of the parties to that in the intermediary's revenue is given by

$$I^{\alpha,\beta} = \frac{\overline{\rho}_X + \overline{\rho}_Y}{1 - \alpha\overline{\rho}_Y - \beta\overline{\rho}_X},$$

where $\overline{\rho}_X$ and $\overline{\rho}_Y$ denote the average pass-through rates:

$$\overline{\rho}_X = \frac{\int_{\underline{y}}^{\overline{y}} \rho_X(y) D_X(p_X(y;\tau);y) g_Y(y)\,\mathrm{d}y}{\int_{\underline{y}}^{\overline{y}} D_X(p_X(y;\tau);y) g_Y(y)\,\mathrm{d}y},$$

$$\overline{\rho}_Y = \frac{\int_{\underline{x}}^{\overline{x}} \rho_Y(x) D_Y(p_Y(x;\tau);x) g_X(x)\,\mathrm{d}x}{\int_{\underline{x}}^{\overline{x}} D_Y(p_Y(x;\tau);x) g_X(x)\,\mathrm{d}x}.$$

With a profit-maximizing intermediary, the incidence is $I^{0,0} = \overline{\rho}_X + \overline{\rho}_Y$, whereas it is $I^{1,1} = (\overline{\rho}_X + \overline{\rho}_Y) / [1 - (\overline{\rho}_X + \overline{\rho}_Y)]$ when the intermediary maximizes total surplus from trade. By denoting $\overline{\rho} = \overline{\rho}_X + \overline{\rho}_Y$, we obtain the same formulae as Weyl and Fabinger (2013) for monopoly and perfect competition under complete information.

The fourth principle of tax incidence explains the factors that determine the pass-through rates. In perfectly competitive markets these factors are the elasticities of demand and supply. Under monopoly also the curvature of the demand matters. In particular, when the demand is logarithmically concave, the monopolistic pass-through rate is less than one, whereas it is more than one if the demand is logarithmically convex.

To revisit the fourth principle, define the elasticity and curvature of the demand function:

$$\epsilon_{D_Y}(p_Y;x) = -\frac{p_Y \frac{\partial D_Y}{\partial p_Y}(p_Y;x)}{D_Y(p_Y;x)} \text{ and } \kappa_{D_Y}(p_Y;x) = -\frac{p_Y \frac{\partial^2 D_Y}{\partial p_Y^2}(p_Y;x)}{\frac{\partial D_Y}{\partial p_Y}(p_Y;x)}.$$

The demand is logarithmically concave if $\kappa_{D_Y} < \epsilon_{D_Y}$ and logarithmically convex if $\kappa_{D_Y} > \epsilon_{D_Y}$. Adopting similar definitions for the other party, we obtain the following result, where the elasticities and curvatures are evaluated at the optimal transaction prices and the other terms at the optimal marginal types:

Proposition 12. *The pass-through rates are given by:*

$$\rho_Y = \frac{1}{1 + \frac{\partial V_X^\alpha}{\partial y} / \frac{\partial v_Y}{\partial y} + (1-\beta)\left(1 - \frac{\kappa_{D_Y}}{\epsilon_{D_Y}}\right)},$$

$$\rho_X = \frac{1}{1 + \frac{\partial V_Y^\beta}{\partial x} / \frac{\partial v_X}{\partial x} + (1-\alpha)\left(1 - \frac{\kappa_{D_X}}{\epsilon_{D_X}}\right)}.$$

Proof. See the Appendix. □

The pass-through rates turn on logarithmic concavity as in Weyl and Fabinger (2013). With private values, the terms $\frac{\partial V_X^\alpha}{\partial y}$ and $\frac{\partial V_Y^\beta}{\partial x}$ are zero. Then, the pass-through to one party is less than one-to-one when its demand is log-concave and more than one-to-one when the demand is log-convex. Furthermore, the pass-through rate is closer to one when the party has more bargaining power. When the values are interdependent, the pass-through also accounts for the change in the marginal virtual valuation of the other party.

Finally, the fifth and the final principle of tax incidence concerns global incidence. As in Weyl and Fabinger (2013), these are obtained by integrating quantity-weighted pass-through rates over the range of the tax change.

5.4 Platform pricing in two-sided markets

In the monopoly model of Rochet and Tirole (2003) a platform posts non-discriminatory transaction prices, and after observing these prices, users on two sides of the market decide whether or not to join the platform. Each side $i = X, Y$ populates a unit mass of potential users who are heterogeneous in how much they benefit from interacting with the other side. By joining the platform, individual users of types $x \in \mathscr{X}$ and

$y \in \mathscr{Y}$ obtain utilities $(x - p_X) N_Y$ and $(y - p_Y) N_X$, where N_i is side i's participation rate and p_i the per-transaction price charged to those users. Types are distributed as before.

By not joining the platform, an individual user obtains zero utility. Therefore, it is a dominant strategy for each user to participate if and only if its type equals or is higher than the transaction price chosen by the platform. Hence, all user pairs (x, y) with $x \geq p_X$ and $y \geq p_Y$ transact with one another and the total volume of transactions is given by

$$N_X N_Y = [1 - G_X(p_X)][1 - G_Y(p_Y)].$$

From each transaction, the platform obtains the total price $p_X + p_Y$ and incurs a cost c. Multiplying the profit margin by the total volume of transactions yields profit

$$\pi(p_X, p_Y) = (p_X + p_Y - c)[1 - G_X(p_X)][1 - G_Y(p_Y)].$$

As Rochet and Tirole (2003) show, monopoly pricing is characterized by the Lerner formulae:

$$\begin{aligned} \frac{p_X - (c - p_Y)}{p_X} &= \frac{1}{\eta_X(p_X)}, \\ \frac{p_Y - (c - p_X)}{p_Y} &= \frac{1}{\eta_Y(p_Y)}, \end{aligned}$$

where η_i denotes the demand elasticity on side i, measuring the percentage change in the user demand when the transaction price p_i is increased by one percent.

Compared to one-sided monopoly pricing, platform pricing differs in the sense that for each transaction the platform receives two prices instead of one. This changes the effective cost of transaction: the optimal monopoly price for one side is chosen based on the transaction cost less the price received from the other side. The optimal price structure equalizes the platform's market power over the two sides: the profit-maximizing linear prices are such that the price ratio equals the ratio of the demand elasticities.

However, the platform can do better by using price mechanisms, where each user on each side sets the transaction price that users on the other side must pay to complete a transaction with that user. We have the following result:

Proposition 13. *Optimal platform pricing is characterized by the Lerner formulae:*

$$\frac{p_X^m(y) - c + \gamma_Y^\beta(y)}{p_X^m(y)} = \frac{1-\alpha}{\eta_X\left(p_X^m(y)\right)},$$
$$\frac{p_Y^m(x) - c + \gamma_X^\alpha(x)}{p_Y^m(x)} = \frac{1-\beta}{\eta_Y\left(p_Y^m(x)\right)},$$

where α and β denote the bargaining weights of sides X and Y, respectively. In particular, $\alpha = \beta = 0$ characterizes monopoly pricing.

Proof. Follows from Proposition (9) and rearranging the equation (2) in Section 5.2. □

The difference to linear pricing is that the optimal price charged to side X users depends on y. This price is chosen based on the cost of transaction less the weighted virtual transaction value for y; that is, the true value less the cost of information. Hence, there is no distortion at the top: the monopoly price charged to side X for transactions with the highest type $\overline{y}$ equals the one-sided monopoly price:

$$\frac{p_X^m(\overline{y}) - (c - \overline{y})}{p_X^m(\overline{y})} = \frac{1-\alpha}{\eta_X\left(p_X^m(\overline{y})\right)}.$$

In particular, this monopoly price is the lowest price charged to side X users. The user with the highest transaction value has an incentive to choose this price in order to maximize the number of transactions with the other side of the market. In essence, the number of transactions determines the quality of the platform for a given user, hence the resemblance to the canonical screening model of Mussa and Rosen (1978) for one-sided markets.

6 Conclusion

This paper uses a mechanism design approach to answer the question of when prices are sufficiently informative for making a decision about a transaction between two privately informed parties. When the values from the transaction are negatively interdependent, as is typically the case for trade between a buyer and a seller, prices indeed suffice to capture all the relevant information. By contrast, this is not true for positively interdependent values. The implementation result extends to private values when the marginal type for one party varies continuously with the type of the other party.

Applying the results on implementation through price mechanisms, we shed new light on existing results and also generate new insights. First, under negatively interdependent values, the equilibrium transaction prices must be strictly decreasing to achieve ex post incentive compatibility. This implies that the set of ex post incentive compatible, ex post individually rational and budget balanced mechanisms is essentially empty. Second, we show that, the equivalence between Bayesian and dominant strategy incentive compatibility boils down to price mechanisms. Third, we extend the five principles of tax incidence to ex post incentive compatible trading mechanisms. Fourth, we characterize optimal platform pricing in two-sided markets.

An important avenue for future research is to extend the analysis to the trade of multiple items. This would require allowing the parties to offer inverse demand functions to each other. Another important extension concerns environments with more than two parties. This may not be straightforward, however, because the equilibrium transaction price paid by one party generally depends on the types of all the other parties. The equilibrium prices must thus summarize price messages chosen by several agents. Finally, it would be interesting to study price mechanisms in the context of an informed intermediary.

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Appendix

Proof of Proposition 3

Proof. By Lemma 1, any weakly ex post implementable allocation (q,t) satisfies the two-part tariff structure. Define functions $\hat{p}_X : \mathscr{Y} \longrightarrow \mathbb{R}$ and $\hat{p}_Y : \mathscr{X} \longrightarrow \mathbb{R}$ by

$$\hat{p}_X(y) = \begin{cases} \overline{x} + p_Y(\overline{x}) - y & \text{if } y < p_Y(\overline{x}), \\ p_X(y) & \text{if } y \in [p_Y(\overline{x}), p_Y(\underline{x})], \\ \underline{x} + p_Y(\underline{x}) - y & \text{if } y > p_Y(\underline{x}), \end{cases}$$

and

$$\hat{p}_Y(x) = \begin{cases} \overline{y} + p_X(\overline{y}) - x & \text{if } x < p_X(\overline{y}), \\ p_Y(x) & \text{if } x \in \left[p_X(\overline{y}), p_X\left(\underline{y}\right)\right], \\ \underline{y} + p_X\left(\underline{y}\right) - x & \text{if } x > p_X\left(\underline{y}\right). \end{cases}$$

Furthermore, define $\hat{f}_X : \mathscr{Y} \longrightarrow \mathbb{R}$ and $\hat{f}_Y : \mathscr{X} \longrightarrow \mathbb{R}$ by

$$\hat{f}_X(y) = \begin{cases} f_X(y) + \underline{x} - \hat{p}_X(y) & \text{if } \hat{p}_X(y) < \underline{x}, \\ f_X(y) & \text{if } \hat{p}_X(y) \geq \underline{x}, \end{cases}$$

and

$$\hat{f}_Y(x) = \begin{cases} f_Y(x) + \underline{y} - \hat{p}_Y(x) & \text{if } \hat{p}_Y(x) < \underline{y}, \\ f_Y(x) & \text{if } \hat{p}_Y(x) \geq \underline{y}. \end{cases}$$

By Lemma 2, the functions $\hat{p}_X$ and $\hat{p}_Y$ are strictly decreasing and continuous, implying that there exist one-to-one mappings $\gamma : \hat{p}_X(\mathscr{Y}) \longrightarrow \mathscr{Y}$ and $\chi : \hat{p}_Y(\mathscr{X}) \longrightarrow \mathscr{X}$, such that $\chi(\hat{p}_Y(x)) = x$ for each $x \in \mathscr{X}$ and $\gamma(\hat{p}_X(y)) = y$ for each $y \in \mathscr{Y}$. We can thus construct a price mechanism $(\mathscr{P}, \mathscr{F}, Q)$ with price menus $\mathscr{P} = \hat{p}_Y(\mathscr{X}) \times \hat{p}_X(\mathscr{Y})$, fee menus $\mathscr{F} = \hat{f}_Y(\mathscr{X}) \times \hat{f}_X(\mathscr{Y})$ and a trade decision, such that for every $p_Y \in \hat{p}_Y(\mathscr{X})$ and $p_X \in \hat{p}_X(\mathscr{Y})$:

$$Q(p_Y, p_X) = q(\chi(p_Y), \gamma(p_X)).$$

For any $x \in \mathscr{X}$ and $y \in \mathscr{Y}$ we then have $Q(\hat{p}_Y(x), \hat{p}_X(y)) = q(x, y)$. Moreover, by construction:

$$\begin{aligned} &\hat{p}_X(y) < \underline{x} \Longrightarrow q(x,y) = 1 \text{ and } p_X(y) = \underline{x}, \\ &\hat{p}_X(y) \geq \underline{x} \Longrightarrow \text{either } q(x,y) = 0 \text{ or } \hat{p}_X(y) = p_X(y). \end{aligned}$$

Therefore:

$$\begin{aligned} t_X(x,y) &= f_X(y) + p_X(y)\, q(x,y) \\ &= f_X(y) + \hat{p}_X(y)\, q(x,y) + \begin{cases} \underline{x} - \hat{p}_X(y) & \text{if } \hat{p}_X(y) < \underline{x}, \\ 0 & \text{if } \hat{p}_X(y) \geq \underline{x}, \end{cases} \\ &= \hat{f}_X(y) + \hat{p}_X(y)\, q(x,y). \end{aligned}$$

Similarly, $t_Y(x,y) = \hat{f}_Y(x) + \hat{p}_Y(x)\, q(x,y)$. Hence, if the allocation has continuous marginal types and is weakly dominant strategy implementable, then there exists a price mechanism that implements it in weakly dominant strategies. $\square$

Proof of Lemma 4

Proof. Suppose the allocation (q,t) is weakly ex post implementable and determined by the transaction values. Let us show that, for any $x, \hat{x} \in \mathscr{X}$, such that $\phi_X(\hat{x}) > \phi_X(x)$, we have $p_Y(x) > p_Y(\hat{x})$. Suppose, by contradiction, that $p_Y(x) \leq p_Y(\hat{x})$. By Lemma

1, there exist $y, \hat{y} \in \mathscr{Y}$, such that:

$$\begin{aligned} v_Y(x,y) &= \phi_Y(x) + \psi_Y(y) \\ &= p_Y(x) \leq p_Y(\hat{x}) \\ &= \phi_Y(\hat{x}) + \psi_Y(\hat{y}) = v_Y(\hat{x},\hat{y}). \end{aligned} \tag{6}$$

By negative interdependence, $\phi_X(\hat{x}) > \phi_X(x)$ implies $\phi_Y(\hat{x}) < \phi_Y(x)$. Then, (6) implies $\psi_Y(y) < \psi_Y(\hat{y})$. By continuity and connectedness, there exists $\tilde{y} \in \mathscr{Y}$, such that $\psi_Y(\tilde{y}) \in (\psi_Y(y), \psi_Y(\hat{y}))$. By (6) we have $v_Y(\hat{x},\tilde{y}) < p_Y(\hat{x})$ and $p_Y(x) < v_Y(x,\tilde{y})$. By Lemma 1

$$\begin{aligned} v_Y(x,\tilde{y}) > p_Y(x) &\Longrightarrow q(x,\tilde{y}) = 1 \Longrightarrow v_X(x,\tilde{y}) \geq p_X(\tilde{y}), \\ v_Y(\hat{x},\tilde{y}) < p_Y(\hat{x}) &\Longrightarrow q(\hat{x},\tilde{y}) = 0 \Longrightarrow v_X(\hat{x},\tilde{y}) \leq p_X(\tilde{y}), \end{aligned}$$

implying $v_X(x,\tilde{y}) \geq v_X(\hat{x},\tilde{y})$. But then $\phi_X(x) \geq \phi_X(\hat{x})$, which contradicts $\phi_X(\hat{x}) > \phi_X(x)$. Therefore, $p_Y(x) > p_Y(\hat{x})$ must hold. This means that $p_Y(x) = p_Y(\hat{x})$ implies $\phi_X(\hat{x}) = \phi_X(x)$ and by negative interdependence $\phi_Y(\hat{x}) = \phi_Y(x)$. Thus, for any $y \in \mathscr{Y}$:

$$p_Y(x) = p_Y(\hat{x}) \Longrightarrow v_X(x,y) = v_X(\hat{x},y) \text{ and } v_Y(x,y) = v_Y(\hat{x},y).$$

Repeating the same argument for the other party, we may conclude that, for any $x, \hat{x} \in \mathscr{X}$ and $y, \hat{y} \in \mathscr{Y}$:

$$\left.\begin{array}{c} p_X(y) = p_X(\hat{y}) \\ p_Y(x) = p_Y(\hat{x}) \end{array}\right\} \Longrightarrow \left.\begin{array}{c} v_X(x,y) = v_X(\hat{x},\hat{y}) \\ v_Y(x,y) = v_Y(\hat{x},\hat{y}) \end{array}\right\}.$$

$\square$

Proof of Lemma 5

Proof. Assume the allocation (q,t) is strongly ex post implementable. Then, by definition, there exists a mechanism $\langle \mathscr{M}, (Q,T)\rangle$, such that for every ex post equilibrium s^* of the game: $(q,t) = (Q,T) \circ s^*$. Furthermore, ex post equilibria exist. As weak ex post implementation is necessary for strong ex post implementation, (q,t) can be written as two-part tariffs according to Lemma 1. Fix any $x, \hat{x} \in \mathscr{X}$, such that $p_Y(\hat{x}) = p_Y(x)$. Suppose, by contradiction, that

$$\tilde{\mathscr{Y}} = \{y \in \mathscr{Y} : q(\hat{x}, y) \neq q(x, y)\}$$

is nonempty and, without loss, consider any $\tilde{y} \in \tilde{\mathscr{Y}}$ such that $q(\hat{x}, \tilde{y}) = 1$ and $q(x, \tilde{y}) = 0$. Then, by Lemma 1:

$$\begin{aligned} q(x, \tilde{y}) = 0 &\Longrightarrow v_X(x, \tilde{y}) \leq p_X(\tilde{y}) \text{ and } v_Y(x, \tilde{y}) \leq p_Y(x), \\ q(\hat{x}, \tilde{y}) = 1 &\Longrightarrow v_X(\hat{x}, \tilde{y}) \geq p_X(\tilde{y}) \text{ and } v_Y(\hat{x}, \tilde{y}) \geq p_Y(\hat{x}). \end{aligned}$$

Using $p_Y(\hat{x}) = p_Y(x)$ we have

$$v_Y(\hat{x}, \tilde{y}) \geq p_Y(\hat{x}) = p_Y(x) \geq v_Y(x, \tilde{y})$$

and

$$v_X(\hat{x}, \tilde{y}) \geq p_X(\tilde{y}) \geq v_X(x, \tilde{y}),$$

which by negative interdependence imply that

$$v_Y(\hat{x},\tilde{y}) = v_Y(x,\tilde{y}),$$
$$v_X(\hat{x},\tilde{y}) = v_X(x,\tilde{y}).$$

Thus, for every $y \in \mathscr{Y}$:

$$\begin{aligned} y \in \tilde{\mathscr{Y}} &\Longrightarrow v_Y(\hat{x},y) = p_Y(\hat{x}) = p_Y(x) = v_Y(x,y) \text{ and} \\ &\quad v_X(\hat{x},y) = p_X(y) = v_X(x,y), \\ y \notin \tilde{\mathscr{Y}} &\Longrightarrow q(\hat{x},y) = q(x,y), \end{aligned} \tag{7}$$

where the latter implication is by definition of $\tilde{\mathscr{Y}}$. Denoting $a = (q,t)$ to shorten notation, for every $y \in \mathscr{Y}$, we have:

$$\begin{aligned} u_X(a(\hat{x},y);x,y) &= [v_X(x,y) - p_X(y)]\, q(\hat{x},y) + f_X(y) \\ &= [v_X(x,y) - p_X(y)]\, q(x,y) + f_X(y) \\ &= u_X(a(x,y);x,y). \end{aligned}$$

Using this equality, we can show that, if party Y plays the equilibrium strategy, party X has a best response to mimic $\hat{x}$ at true type x. Indeed, for any ex post equilibrium s^* in $\langle \mathscr{M}, A \rangle$, where we denote $A = (Q,T)$, we have that for any $m_X \in \mathscr{M}_X$ and $y \in \mathscr{Y}$:

$$\begin{aligned} u_X(A(s_X^*(\hat{x}), s_Y^*(y));x,y) &= u_X(a(\hat{x},y);x,y) \\ &= u_X(a(x,y);x,y) \\ &= u_X(A(s_X^*(x), s_Y^*(y));x,y) \\ &\geq u_X(A(m_X, s_Y^*(y));x,y). \end{aligned}$$

As $\tilde{\mathscr{Y}}$ is not empty by hypothesis, by strong ex post implementability, party X mimicking $\hat{x}$ at true type x does not constitute an ex post equilibrium. Therefore, there exist $y \in \mathscr{Y}$ and $m_Y \in \mathscr{M}_Y$, such that Y will deviate:

$$u_Y\left(A\left(s_X^*\left(\hat{x}\right), s_Y^*\left(y\right)\right); x, y\right) < u_Y\left(A\left(s_X^*\left(\hat{x}\right), m_Y\right); x, y\right).$$

Furthermore, by the definition of an ex post equilibrium:

$$u_Y\left(A\left(s_X^*\left(\hat{x}\right), s_Y^*\left(y\right)\right); \hat{x}, y\right) \geq u_Y\left(A\left(s_X^*\left(\hat{x}\right), m_Y\right); \hat{x}, y\right).$$

Together these two inequalities imply $v_Y\left(x, y\right) \neq v_Y\left(\hat{x}, y\right)$. But then, by the separability assumption, $v_Y\left(x, y\right) \neq v_Y\left(\hat{x}, y\right)$ must be true for all $y \in \mathscr{Y}$, implying by (7) that $\tilde{\mathscr{Y}}$ is empty; a contradiction. Thus, $p_Y\left(\hat{x}\right) = p_Y\left(x\right)$ implies that, for all $y \in \mathscr{Y}$ we have $q\left(\hat{x}, y\right) = q\left(x, y\right)$. This in turn implies $f_Y\left(\hat{x}\right) = f_Y\left(x\right)$ by strong ex post implementability. Repeating the same argument for the other side, we conclude that for any $x, \hat{x} \in \mathscr{X}$ and $y, \hat{y} \in \mathscr{Y}$:

$$\begin{aligned} p_X\left(\hat{y}\right) = p_X\left(y\right) &\Longrightarrow q\left(x, \hat{y}\right) = q\left(x, y\right) \text{ and } f_X\left(\hat{y}\right) = f_X\left(y\right), \\ p_Y\left(\hat{x}\right) = p_Y\left(x\right) &\Longrightarrow q\left(\hat{x}, y\right) = q\left(x, y\right) \text{ and } f_Y\left(\hat{x}\right) = f_Y\left(x\right). \end{aligned}$$

□

Proof of Proposition 8

Proof. We will divide the proof into three cases, depending on the trade rule:

Case 1. The transaction is always completed: $q\left(x, y\right) = 1$ *for all* $\left(x, y\right) \in \mathscr{X} \times \mathscr{Y}$. By Lemma 1, the allocation is weakly ex post implementable iff $p_Y\left(x\right) = v_Y\left(x, \underline{y}\right)$ and $p_X\left(y\right) = v_X\left(\underline{x}, y\right)$ for each $\left(x, y\right) \in \mathscr{X} \times \mathscr{Y}$, with strictly decreasing transaction price functions. Together with ex post budget balance strict monotonicity implies:

$$t_X(x,y) = p_X(y) + f_X(y) = 0 \Longrightarrow v_X(\underline{x}, y) = p_X(y) = -f_X(y),$$
$$t_Y(x,y) = p_Y(x) + f_Y(x) = 0 \Longrightarrow v_Y\left(x, \underline{y}\right) = p_Y(x) = -f_Y(x),$$

where $f_X(y) \leq 0$ and $f_Y(x) \leq 0$ by ex post participation.

Case 2: The transaction is never completed: $q(x,y) = 0$ *for all* $(x,y) \in \mathscr{X} \times \mathscr{Y}$. Then, by Lemma 1, $p_Y(x) = v_Y(x, \overline{y})$ and $p_X(y) = v_X(\overline{x}, y)$. By ex post participation and budget balance:

$$t_X(x,y) = f_X(y) = 0,$$
$$t_Y(x,y) = f_Y(x) = 0.$$

Case 3: $q(x,y) = 1$ *for some* $(x,y) \in \mathscr{X} \times \mathscr{Y}$ *and* $q(x,y) = 0$ *for others.* Then $f_X(y) = f_Y(x) = 0$ by ex post participation and budget balance, which also imply

$$q(x,y) = 1 \Longrightarrow p_Y(x) + p_X(y) = 0.$$

By Lemma 3 we must then have

$$q(x,y) = \begin{cases} 1 & \text{if } x = \overline{x} \text{ and } y = \overline{y}, \\ 0 & \text{otherwise,} \end{cases}$$

which by Lemma 1 implies that $p_Y(x) = v_Y(\overline{x}, \overline{y})$ and $p_X(\overline{y}) = v_X(\overline{x}, \overline{y})$. Therefore

$$v_Y(\overline{x}, \overline{y}) + v_X(\overline{x}, \overline{y}) = 0.$$

$\square$

Proof of Proposition 9

Proof. By Theorem 1 in Gershkov et al. (2013), we may replace the Bayesian incentive compatibility constraint with dominant strategy incentive compatibility. By Lemma 1 the optimal allocation must then satisfy the two-part tariff structure. We can write:

$$\begin{aligned}\bar{u}_X(x) &= \int_{\underline{y}}^{\overline{y}} [x - p_X(y)]\, q(x,y)\, g_Y(y)\,\mathrm{d}y - \int_{\underline{y}}^{\overline{y}} f_X(y)\, g_Y(y)\,\mathrm{d}y \\ &= \int_{p_Y(x)}^{\overline{y}} [x - p_X(y)]\, g_Y(y)\,\mathrm{d}y - \int_{\underline{y}}^{\overline{y}} f_X(y)\, g_Y(y)\,\mathrm{d}y\end{aligned}$$

where $p_X(y) \geq \underline{x}$ and therefore

$$\bar{u}_X(\underline{x}) \leq -\int_{\underline{y}}^{\overline{y}} f_X(y)\, g_Y(y)\,\mathrm{d}y.$$

By ex interim individual rationality, the average fixed fee cannot be positive. It is thus optimal to choose zero fixed fees.

Taking the expectation over x we obtain:

$$\begin{aligned}\int_{\underline{x}}^{\overline{x}} \bar{u}_X(x)\, g_X(x)\,\mathrm{d}x &= \int_{\underline{x}}^{\overline{x}} \int_{p_Y(x)}^{\overline{y}} [x - p_X(y)]\, g_Y(y)\, g_X(x)\,\mathrm{d}y\mathrm{d}x \\ &= \int_{\underline{y}}^{\overline{y}} \int_{p_X(y)}^{\overline{x}} [x - p_X(y)]\, g_X(x)\, g_Y(y)\,\mathrm{d}x\mathrm{d}y,\end{aligned}$$

where the second equality follows from Lemma 1 and Fubini's theorem, allowing us to switch the order of integration. Using integration by parts:

$$\begin{aligned}\int_{\underline{x}}^{\overline{x}} \bar{u}_X(x)\, g_X(x)\,\mathrm{d}x &= \int_{\underline{y}}^{\overline{y}} \int_{p_X(y)}^{\overline{x}} [1 - G_X(x)]\, g_Y(y)\,\mathrm{d}x\mathrm{d}y \\ &= \int_{\underline{x}}^{\overline{x}} [1 - G_X(x)]\, [1 - G_Y(p_Y(x))]\,\mathrm{d}x,\end{aligned}$$

where the second equality follows from switching the order of integration again. Using the fact that the revenue from intermediation can be written as total welfare less the utilities of the parties, we get

$$W^{\alpha,\beta} = \int_{\underline{x}}^{\overline{x}} \int_{p_Y(x)}^{\overline{y}} \left[\gamma_X^{\alpha}(x) + \gamma_Y^{\beta}(y)\right] g_Y(y)\, g_X(x)\, \mathrm{d}y \mathrm{d}x.$$

By the assumption that the densities are continuous, the weighted virtual valuations are continuous. As they are also strictly increasing, the optimal transaction price equals $\max\left\{\min\left\{\overline{y}, p_Y^m(x)\right\} \underline{y}\right\}$, where $p_Y^m(x)$ satisfies

$$\gamma_X^{\alpha}(x) + \gamma_Y^{\beta}\left(p_Y^m(x)\right) = 0,$$

being continuous and strictly decreasing. Hence, we can apply Proposition 3 to conclude. □

Proof of Proposition 10

Proof. Fix any (τ_X, τ_Y, τ_I) and any weakly ex post implementable allocation (q, t), and let $v_Y^{\tau}(x, y) \coloneqq v_Y(x, y) - \tau_Y$ and $v_X^{\tau}(x, y) \coloneqq v_X(x, y) - \tau_X$ denote the net transaction values. By Lemma 1, there exist transaction prices and fixed fees such that, for every $x \in \mathscr{X}$ and $y \in \mathscr{Y}$, the trade rule satisfies:

$$q(x, y) = 0 \Longrightarrow v_Y^{\tau}(x, y) \leq p_Y(x) \text{ and } v_X^{\tau}(x, y) \leq p_X(y),$$
$$q(x, y) = 1 \Longrightarrow v_Y^{\tau}(x, y) \geq p_Y(x) \text{ and } v_X^{\tau}(x, y) \geq p_X(y),$$

and the payments satisfy:

$$t_Y(x,y) = f_Y(x) + p_Y(x)\, q(x,y),$$
$$t_X(x,y) = f_X(y) + p_X(y)\, q(x,y).$$

Consider now any $(\tilde{\tau}_X, \tilde{\tau}_Y, \tilde{\tau}_I)$ with $\tilde{\tau}_X + \tilde{\tau}_Y + \tilde{\tau}_I = \tau_X + \tau_Y + \tau_I$, and the associated net transaction values $v_Y^{\tilde{\tau}}(x,y)$ and $v_X^{\tilde{\tau}}(x,y)$. Let $\tilde{p}_Y(\cdot) = p_Y(\cdot) + \tau_Y - \tilde{\tau}_Y$ and $\tilde{p}_X(\cdot) = p_X(\cdot) + \tau_X - \tilde{\tau}_X$, and consider the payment rule $\tilde{t}$ such that, for every $x \in \mathscr{X}$ and $y \in \mathscr{Y}$:

$$\tilde{t}_Y(x,y) = f_Y(x) + \tilde{p}_Y(x)\, q(x,y),$$
$$\tilde{t}_X(x,y) = f_X(y) + \tilde{p}_X(x)\, q(x,y).$$

By construction, for any $x \in \mathscr{X}$ and $y \in \mathscr{Y}$, $v_Y^{\tau}(x,y) \leq p_Y(x)$ if and only if $v_Y^{\tilde{\tau}}(x,y) \leq \tilde{p}_Y(x)$ and $v_X^{\tau}(x,y) \leq p_X(y)$ if and only if $v_X^{\tilde{\tau}}(x,y) \leq \tilde{p}_X(y)$. Therefore the allocation $\left(q, \tilde{t}\right)$ is weakly ex post implementable. By construction, these payments deliver the same net payoffs as the original ones:

$$\begin{aligned} v_i^{\tau}(x,y)\, q(x,y) - \tilde{t}_i(x,y) &= \left[v_i(x,y) - \tilde{\tau}_i\right] q(x,y) - \tilde{t}_i(x,y) \\ &= \left[v_i(x,y) - \tau_i\right] q(x,y) + \left(\tau_i - \tilde{\tau}_i\right) q(x,y) - \tilde{t}_i(x,y) \\ &= \left[v_i(x,y) - \tau_i\right] q(x,y) - t_i(x,y) \\ &= v_i^{\tilde{\tau}}(x,y)\, q(x,y) - t_i(x,y) \end{aligned}$$

and

$$
\begin{aligned}
\tilde{t}_Y(x,y)+\tilde{t}_X(x,y)-\tilde{\tau}_I q(x,y) &= \tilde{t}_Y(x,y)+\tilde{t}_X(x,y)-\left(\tau_Y-\tilde{\tau}_Y+\tau_X-\tilde{\tau}_X\right)q(x,y)\\
&\quad+\left(\tau_Y-\tilde{\tau}_Y+\tau_X-\tilde{\tau}_X-\tilde{\tau}_I\right)q(x,y)\\
&= t_X(x,y)+t_Y(x,y)-\tau_I q(x,y).
\end{aligned}
$$

□

Proof of Proposition 12

Proof. The optimal transaction price $p_Y(x;\tau)=v_Y(x,y^q(x;\tau))$ is characterized by

$$
V^{\alpha,\beta}(x,y^q(x;\tau))=V_X^{\alpha}(x,y^q(x;\tau))+V_Y^{\beta}(x,y^q(x;\tau))=\tau,
$$

where

$$
V_Y^{\beta}(x,y^q(x;\tau))=v_Y(x,y^q(x;\tau))-(1-\beta)\frac{\partial v_Y(x,y^q(x;\tau))}{\partial y}\frac{1-G_Y(y^q(x;\tau))}{g_Y(y^q(x;\tau))}.
$$

To express this in terms of the transaction price and the demand, recall that $\hat{y}(p_Y;x)$ is defined by $p_Y=v_Y(x,\hat{y}(p_Y;x))$ and the demand by $D_Y(p_Y;x)=1-G_Y(\hat{y}(p_Y;x))$. We thus have:

$$
\frac{\partial D_Y(p_Y;x)}{\partial p_Y}=-g_Y(\hat{y}(p_Y;x))\frac{\partial \hat{y}(p_Y;x)}{\partial p_Y}=-\frac{g_Y(\hat{y}(p_Y;x))}{\frac{\partial v_Y}{\partial y}(x,\hat{y}(p_Y;x))}.
$$

Therefore:

$$
\frac{\partial v_Y(x,\hat{y}(p_Y;x))}{\partial y}\frac{1-G_Y(\hat{y}(p_Y;x))}{g_Y(\hat{y}(p_Y;x))}=-\frac{D_Y(p_Y;x)}{\frac{\partial D_Y}{\partial p_Y}(p_Y;x)}.
$$

By the identity $p_Y(x;\tau) = v_Y(x, y^q(x;\tau))$, we have $\hat{y}(p_Y(x;\tau);x) = y^q(x;\tau)$ and therefore

$$V_Y^{\beta}(x, y^q(x;\tau)) = p_Y(x;\tau) + (1-\beta)\frac{D_Y(p_Y(x;\tau);x)}{\frac{\partial D_Y}{\partial p_Y}(p_Y(x;\tau);x)}.$$

The optimality condition can thus be written as

$$p_Y(x;\tau) = \tau - V_X^{\alpha}(x, \hat{y}(p_Y(x;\tau);x)) - (1-\beta)\frac{D_Y(p_Y(x;\tau);x)}{\frac{\partial D_Y}{\partial p_Y}(p_Y(x;\tau);x)}.$$

By taking the derivative with respect to τ we obtain:

$$\begin{aligned}\frac{\partial p_Y(x;\tau)}{\partial \tau} &= 1 - \frac{\frac{\partial V_X^{\alpha}}{\partial y}(x, \hat{y}(p_Y(x;\tau);x))}{\frac{\partial v_Y}{\partial y}(x, \hat{y}(p_Y(x;\tau);x))}\frac{\partial p_Y(x;\tau)}{\partial \tau} \\ &\quad - (1-\beta)\left[1 - \frac{\frac{\partial^2 D_Y}{\partial p_Y^2}(p_Y(x;\tau);x)\, D_Y(p_Y(x;\tau);x)}{\frac{\partial D_Y}{\partial p_Y}(p_Y(x;\tau);x)\frac{\partial D_Y}{\partial p_Y}(p_Y(x;\tau);x)}\right]\frac{\partial p_Y(x;\tau)}{\partial \tau}.\end{aligned}$$

The term in the brackets can then be expressed using the definitions of curvature and elasticity of the demand:

$$\epsilon_{D_Y}(p_Y;x) = -\frac{p_Y\frac{\partial D_Y}{\partial p_Y}(p_Y;x)}{D_Y(p_Y;x)} \text{ and } \kappa_{D_Y}(p_Y;x) = -\frac{p_Y\frac{\partial^2 D_Y}{\partial p_Y^2}(p_Y;x)}{\frac{\partial D_Y}{\partial p_Y}(p_Y;x)}.$$

Rearranging the equation implies the expression stated in the proposition. □

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